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Charged axially symmetric solution and energy in teleparallel theory equivalent to general relativity

G.G.L. Nashed[a]

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

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Abstract. An exact charged solution with axial symmetry is obtained in the teleparallel equivalent of general relativity. The associated metric has the structure function $G(\xi)=1-\xi^2-2mA\xi^3-q^2A^2\xi^4$. The fourth order nature of the structure function can make calculations cumbersome. Using a coordinate transformation we get a tetrad whose metric has the structure function in a factorizable form $(1-\xi^2)(1+r_+A\xi)(1+r_-A\xi)$ with $r_\pm$ as the horizons of Reissner–Nordström space-time. This new form has the advantage that its roots are now trivial to write down. Then, we study the singularities of this space-time. Using another coordinate transformation, we obtain a tetrad field. Its associated metric yields the Reissner–Nordström black hole. In calculating the energy content of this tetrad field using the gravitational energy-momentum, we find that the resulting form depends on the radial coordinate! Using the regularized expression of the gravitational energy-momentum in the teleparallel equivalent of general relativity we get a consistent value for the energy.

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1 Introduction

The charged C-metric line element and electromagnetic potential are given by [16, 17]

$$\mathrm{d}s^2=\frac{1}{A^2(x-y)^2}\left[G(y)\,\mathrm{d}t^2-\frac{\mathrm{d}y^2}{G(y)}+\frac{\mathrm{d}x^2}{G(x)}+G(x)\,\mathrm{d}\phi^2\right],$$
$$\mathcal{A}=Qy\,\mathrm{d}t\,, \tag{1}$$

where $\mathcal{A}$ is the electromagnetic vector potential, Q is the charge parameter, and the structure function G is defined by

$$G(\xi)\overset{\text{def.}}{=}1-\xi^2-2mA\xi^3-q^2A^2\xi^4. \tag{2}$$

Here m and A are positive parameters related to the mass and acceleration of the black hole, such that $mA<1/\sqrt{27}$. The fact that G is a fourth order polynomial in ξ means that one cannot in general write down a simple expression for its roots. Since these roots play an important role in almost every analysis of the charged C-metric, most results have to be expressed implicitly in terms of them. Any calculation which requires their explicit forms would naturally be very tedious if not impossible to carry out [2–4].

At present, teleparallel theory seems to be popular again. There is a trend of analyzing the basic solutions of general relativity with teleparallel theory and comparing the results. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [5–17] or metric-affine gravity [18]. The physics relevant to geometry may be related to the teleparallel description of gravity [19–23]. Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using *prolongation techniques* (see [24] and references therein). The teleparallel approach is used for the positive-gravitational-energy proof [25]. A relation between spinor Lagrangian and teleparallel theory is established [26]. In the metric-affine generalization of teleparallelism, Obukhov et al. [27–29] have shown that there is an inconsistency in the coupling of spinors. Mielke [30] demonstrated the consistency of the coupling of the Dirac fields to the teleparallel equivalent of general relativity (TEGR). However, Obukhov and Pereira [27–29] have shown that this demonstration is not correct. They also [27–29] have studied the general teleparallel gravity model within the framework of the metric-affine gravity theory. Nester et al. [31–33] have considered the quasilocal center-of-mass (COM) in tetrad teleparallel gravity. They have used the covariant Hamiltonian formalism, in which quasilocal quantities are given by the Hamiltonian boundary term, along with the covariant asymptotic Hamiltonian boundary expressions. Consideration of the COM not only gives the most restrictive asymptotic conditions on the vari-

[a] e-mail: nashed652001@yahoo.com

ables but also gives strong constraints on the acceptable expressions [31–33].

For a satisfactory description of the total energy of an isolated system it is necessary that the energy density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well known that there exists no covariant, non-trivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than being an intrinsic drawback of the theory [34, 35]. Møller has shown that the problem of the energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian space-time [36]. In a series of papers [36–39], he was able to obtain a general expression for a satisfactory energy-momentum complex in the teleparallel space-time. Xu and Jing derived the field equation with a cosmological term and studied the energy of the general 4-dimensional stationary axisymmetric space-time in the context of the Hamiltonian formulation of the TEGR [40].

It is well known that TEGR [19–35] provides an alternative description of Einstein's general relativity. In this theory the gravitational field is described by the tetrad field e^a_μ. In fact the first attempt to construct a theory of the gravitational field in terms of a set of four linearly independent vector fields in the Weitzenböck geometry is due to Einstein [41–43].

A well posed and mathematically consistence expression for the gravitational energy has been developed [35]. It arises in the realm of the Hamiltonian formulation of the TEGR [44] and satisfies several crucial requirements for any acceptable definition of the gravitational energy. The gravitational energy-momentum P^a [35, 45] obtained in the framework of the TEGR has been investigated in the context of several distinct configurations of the gravitational field. For asymptotically flat space-times P^0 yields the ADM energy [46]. In the context of tetrad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition

$$e_{a\mu} \cong \eta_{a\mu} + \frac{1}{2} h_{a\mu}(1/r)\,, \tag{3}$$

and by the condition $\partial_\mu e^a_\mu = O(1/r^2)$ in the asymptotic limit $r \to \infty$, with $\eta_{ab} = (-1, +1, +1, +1)$ the metric of Minkowski space-time. An important property of tetrad fields that satisfy (3) is that in the flat space-time limit one has $e^a_\mu(t, x, y, z) = \delta^a_\mu$, and therefore the torsion tensor $T^a_{\mu\nu} = 0$. Maluf [47] has extended the definition P^a for the gravitational energy-momentum [35, 44] to any arbitrary tetrad fields, i.e., for the tetrad fields that satisfy $T^a_{\mu\nu} \neq 0$ for the flat space-time. The redefinition is the only possible consistent extension of P^a, valid for the tetrad fields that do not satisfy (3).

It is the aim of the present work to derive a charged axially symmetric solution in TEGR. In Sect. 2 we give a brief review of the TEGR of the coupled gravitational and electromagnetic fields. A tetrad having axial symmetry with six unknown functions in x and y is applied to the field equations and a solution of charged axial symmetry is obtained in Sect. 3. A coordinate transformation is applied to the tetrad obtained in Sect. 3, to put the structure function in a factorizable form. The advantage of this transformation is that it makes the roots of the original solution factorizable. Also in Sect. 3, the tetrad singularities (15) (see below) are studied. In Sect. 4, another coordinate transformation is applied to the tetrad (15) and a tetrad so that its associated metric gives the Reissner–Nordström black hole is obtained. The energy content of the tetrad (20) (also see below) is calculated in Sect. 4 using the gravitational energy-momentum [35, 47], and an unsatisfactory value of energy is obtained. In Sect. 5 we use the regularized expression for the gravitational energy-momentum to calculate the energy. A discussion and conclusion of the obtained results are given in the final section[1].

2 The TEGR for gravitation and electromagnetism

In a space-time with absolute parallelism the parallel vector fields $e_a^{\ \mu}$ define the non-symmetric affine connection

$$\Gamma^\lambda_{\mu\nu} \stackrel{\text{def.}}{=} e_a^{\ \lambda} e^a_{\ \mu,\nu}\,, \tag{4}$$

where $e_{a\mu,\nu} = \partial_\nu e_{a\mu}$.[2] The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$ is identically vanishing, however. The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu\,. \tag{5}$$

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[3] [35, 48]

$$\begin{aligned}\mathcal{L}_\text{G} = eL_\text{G} &= -\frac{e}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^a T_a\right) - L_m \\ &= -\frac{e}{16\pi}\Sigma^{abc}T_{abc} - L_m\,,\end{aligned} \tag{6}$$

where $e = \det(e^a_\mu)$. The tensor Σ^{abc} is defined by

$$\Sigma^{abc} \stackrel{\text{def.}}{=} \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c)\,. \tag{7}$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$\begin{aligned}&T^a_{\mu\nu} \stackrel{\text{def.}}{=} e^a_\lambda T^\lambda_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu\,,\\ &T^a_{bc} \stackrel{\text{def.}}{=} e_b^{\ \mu} e_c^{\ \nu} T^a_{\mu\nu}\,, \qquad T^a \stackrel{\text{def.}}{=} T^{ba}_{\ \ b}\,.\end{aligned} \tag{8}$$

[1] The computer algebra system Maple 6 is used in some calculations.

[2] The space-time indices μ, ν, ... and the $SO(3,1)$ indices a, b, ... run from 0 to 3. Time and space indices are indicated by $\mu = 0, i$, and $a = (0), (i)$.

[3] Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

The quadratic combination $\Sigma^{abc}T^{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term [34]. L_{m} represents the Lagrangian density for the matter fields. The electromagnetic Lagrangian density $L_{\mathrm{e.m.}}$ is given by [50]

$$L_{\mathrm{e.m.}} \stackrel{\text{def.}}{=} -\frac{e}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\,, \tag{9}$$

where $F_{\mu\nu}$ is given by[4] $F_{\mu\nu} \stackrel{\text{def.}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and A_μ is the electromagnetic potential.

The gravitational and electromagnetic field equations for the system described by $L_{\mathrm{G}} + L_{\mathrm{e.m.}}$ are the following:

$$e_{a\lambda} e_{b\mu} \partial_\nu (e\Sigma^{b\lambda\nu}) - e\left(\Sigma^{b\nu}_{a} T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}\right) = \frac{1}{2}\kappa e T_{a\mu},$$
$$\partial_\nu (eF^{\mu\nu}) = 0\,, \tag{10}$$

where

$$\frac{\delta L_{\mathrm{m}}}{\delta e^{a\mu}} \equiv e T_{a\mu}\,.$$

It is possible to prove by explicit calculations that the left hand side of the symmetric field equations of (10) is exactly given by [35]

$$\frac{e}{2}\left[R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e)\right].$$

3 Charged axially symmetric solution

In this section we will assume the parallel vector fields to have the form

$$(e_\mu^a) = \begin{pmatrix} A_1(x,y) & 0 & 0 & 0 \\ 0 & B_1(x,y)\cos\phi & 0 & -B_2(x,y)\sin\phi \\ 0 & 0 & C_1(x,y) & 0 \\ 0 & D_1(x,y)\sin\phi & 0 & D_2(x,y)\cos\phi \end{pmatrix}, \tag{11}$$

where $A_1(x,y)$, $B_1(x,y)$, $B_2(x,y)$, $C_1(x,y)$, $D_1(x,y)$ and $D_2(x,y)$ are unknown functions. We use a non-diagonal form of the tetrad given in (11) in spite the metric reproduced being in a diagonal form. This is due to the fact that there are non-diagonal tetrads reproduced, there being a diagonal metric, however, more physics is needed to explain the obtained results [51, 52].

Applying (11) to the field equations (10) we obtain the unknown functions in the form

$$A_1(x,y) = \frac{\sqrt{G(y)}}{A(x-y)}\,, \qquad B_1(x,y) = \frac{1}{A(x-y)\sqrt{G(x)}}\,,$$
$$B_2(x,y) = \frac{\sqrt{G(x)}}{A(x-y)}\,, \qquad C_1(x,y) = \frac{1}{A(x-y)\sqrt{G(y)}}\,,$$
$$D_1(x,y) = \frac{1}{A(x-y)\sqrt{G(x)}}\,,\ D_2(x,y) = \frac{\sqrt{G(x)}}{A(x-y)}\,, \tag{12}$$

where $G(\xi) = 1 - \xi^2 - 2mA\xi^3 - Q^2A^2\xi^4$, and the electromagnetic potential is given by $\mathcal{A} = Qy\,\mathrm{d}t$, where $Q = \frac{q}{2\sqrt{\pi}}$ [49, 50] is the charge parameter. The associated metric of solution (12) has the form (1), which is the charged C-metric. As we have seen in the introduction, in general one cannot easily write down simple expressions of the roots of G. Therefore, one must find some coordinate transformation which makes the roots of G written explicitly and this would in turn simplify a certain analysis of the charged C-metric. This coordinate transformation has the form [53]

$$x = B(\bar{x} - c_1)\,, \quad y = B(\bar{y} - c_1)\,, \quad \phi = B_1\bar{\phi}\,, \quad t = B_1\bar{t}\,, \tag{13}$$

where

$$B = \left(\frac{1 - \bar{A}^2\bar{Q}^2 + 6\bar{m}\bar{A}c_1 + 6\bar{Q}^2\bar{A}^2{c_1}^2}{1 + (1-\bar{Q}^2\bar{A}^2){c_1}^2 + 4\bar{m}\bar{A}{c_1}^3 + 3\bar{Q}^2\bar{A}^2{c_1}^4}\right)^{1/2},$$
$$B_1 = \sqrt{1 - \bar{A}^2\bar{Q}^2 + 6\bar{m}\bar{A}c_1 + 6\bar{Q}^2\bar{A}^2{c_1}^2} \times \sqrt{1 + (1-\bar{Q}^2\bar{A}^2){c_1}^2 + 4\bar{m}\bar{A}{c_1}^3 + 3\bar{Q}^2\bar{A}^2{c_1}^4}\,,$$

where $\bar{x}$, $\bar{y}$, $\bar{\phi}$, $\bar{t}$ are the new coordinates and

$$Q = \frac{\bar{Q}}{1 - \bar{A}^2\bar{Q}^2 + 6\bar{m}\bar{A}c_1 + 6\bar{Q}^2\bar{A}^2{c_1}^2}\,,$$
$$m = \frac{\bar{m} + 2\bar{Q}^2\bar{A}c_1}{\left(1 - \bar{A}^2\bar{Q}^2 + 6\bar{m}\bar{A}c_1 + 6\bar{Q}^2\bar{A}^2{c_1}^2\right)^{3/2}}\,,$$
$$A = \sqrt{1 + (1-\bar{Q}^2\bar{A}^2){c_1}^2 + 4\bar{m}\bar{A}{c_1}^3 + 3\bar{Q}^2\bar{A}^2{c_1}^4}\,\bar{A}\,,$$
$$4\bar{Q}^2\bar{A}^2{c_1}^3 + 6\bar{m}\bar{A}{c_1}^2 + 2(1-\bar{Q}^2\bar{A}^2)c_1 = 2\bar{m}\bar{A}\,. \tag{14}$$

Applying the coordinate transformation (13) to the tetrad (11) with the solution (12), we obtain

$$(e_\mu^a) = \begin{pmatrix} \frac{H(\bar{y})}{\bar{A}(\bar{x}-\bar{y})} & 0 & 0 & 0 \\ 0 & \frac{\cos\bar{\phi}^*}{\bar{A}(\bar{x}-\bar{y})H(\bar{x})} & 0 & -\frac{H(\bar{x})\sin\bar{\phi}^*}{\bar{A}(\bar{x}-\bar{y})} \\ 0 & 0 & \frac{1}{\bar{A}(\bar{x}-\bar{y})H(\bar{y})} & 0 \\ 0 & \frac{\sin\bar{\phi}^*}{\bar{A}(\bar{x}-\bar{y})H(\bar{x})} & 0 & \frac{\cos\bar{\phi}^* H(\bar{x})}{\bar{A}(\bar{x}-\bar{y})} \end{pmatrix}, \tag{15}$$

where

$$\bar{\phi}^* = B_1\bar{\phi}\,,$$

with B_1 given in (13) and

$$H(\xi) = \sqrt{1 - \xi^2 + \bar{Q}^2\bar{A}^2\xi^2 + 2\bar{m}\bar{A}\xi - 2\bar{m}\bar{A}\xi^3 - \bar{Q}^2\bar{A}^2\xi^4}\,.$$

[4] Heaviside–Lorentz rationalized units will be used throughout this paper.

The associated metric of the tetrad field given by (15) is given by

$$\mathrm{d}s^2 = \frac{1}{A^2(x-y)^2} \times \left[G_1(y)\,\mathrm{d}t^2 - \frac{\mathrm{d}y^2}{G_1(y)} + \frac{\mathrm{d}x^2}{G_1(x)} + G_1(x)\,\mathrm{d}\phi^2\right], \tag{16}$$

where $G_1(\xi)$ is defined by

$$G_1(\xi) \stackrel{\text{def.}}{=} (1-\xi^2)(1+r_+A\xi)(1+r_-A\xi) = H^2(\xi)\,,$$

with

$$r_\mp = \bar{m} \mp \sqrt{\bar{m}^2 - \bar{Q}^2}\,,$$

and $0 \le r_-A \le r_+A < 1$ [53], and the electromagnetic potential has the form

$$\mathcal{A} = \bar{Q}(\bar{x} - c_1)\,\mathrm{d}t\,.$$

It is clear from (16) that one can gets the roots easily having the form

$$\xi_{1,2} = -\frac{1}{r_\mp A}\,, \quad \xi_{3,4} = \mp 1\,, \tag{17}$$

which obey

$$\xi_1 \le \xi_2 < \xi_3 < \xi_4\,.$$

Now we are going to study the singularities of the tetrad (15).

In teleparallel theories of gravity we mean by a singularity of space-time [50] the singularity of the scalar concomitants of the curvature and torsion tensors.

Using the definitions of the Riemann–Christoffel, the Ricci tensors, the Ricci scalar, the torsion tensor and the basic vector (8), [54] we obtain for the solution of (15)

$$\begin{aligned} R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} &= F_1(\bar{x},\bar{y})\,,\\ R^{\mu\nu}R_{\mu\nu} &= F_2(\bar{x},\bar{y})\,,\\ R &= F_3(\bar{x},\bar{y})\,, \end{aligned}$$

$$T^{abc}T_{abc} = \frac{F_4(\bar{x},\bar{y})}{(1-\bar{x}^2)(1-\bar{y}^2)(1+2\bar{x}\bar{m}\bar{A}+\bar{Q}^2\bar{x}^2\bar{A}^2)(1+2\bar{y}\bar{m}\bar{A}+\bar{Q}^2\bar{y}^2\bar{A}^2)}\,,$$

$$T^aT_a = \frac{F_5(\bar{x},\bar{y})}{(1-\bar{x}^2)(1-\bar{y}^2)(1+2\bar{x}\bar{m}\bar{A}+\bar{Q}^2\bar{x}^2\bar{A}^2)(1+2\bar{y}\bar{m}\bar{A}+\bar{Q}^2\bar{y}^2\bar{A}^2)}\,, \tag{18}$$

where F_i, $i = 1, \ldots, 5$ are too lengthy functions of $\bar{x}$ and $\bar{y}$. It is clear from (18) that the scalars of the torsion and the basic vector have the same singularities, as the dominators of both are the same. Let us discuss these singularities.

1) If $\bar{x} = \bar{y} = \xi_3$, then all the scalars of (18) have a singularity which is called *asymptotic infinity* [53].
2) When $\bar{y} = \xi_2$, there is a singularity which is called *a black hole event horizon* [53].
3) When $\bar{y} = \xi_3$, there is also a singularity which is the *acceleration horizon*.
4) When $\bar{x} = \xi_4$, there is a singularity which makes *a symmetry axis between event and acceleration horizons*.
5) When $\bar{x} = \xi_3$ there is a singularity which makes *a symmetry axis joining event horizon and asymptotic horizon*.
6) When $\bar{x} = \xi_3$ and $\bar{y} = \xi_4$, there will be *a conical singularity* [53].

4 Energy content

To write the tetrad field given in (15) in spherical polar coordinates, we will use the following coordinate transformation [53]:

$$\bar{x} = \cos\theta\,, \quad \bar{y} = -\frac{1}{\bar{A}r}\,, \quad \bar{\phi}^* = \bar{\phi}^*\,, \quad \bar{t} = \bar{A}t_1\,, \tag{19}$$

where r, θ, t_1 are the new coordinates. Applying the transformation (19) to the tetrad field (15) we get

$$\left(e^a{}_\mu\right) = \begin{pmatrix} -\frac{\sqrt{r^2-2\bar{m}r+\bar{Q}^2}H_1}{rG_2} & 0 & 0 & 0\\ 0 & 0 & \frac{r\cos\bar{\phi}^*}{FG_2} & \frac{rF\sin\theta\sin\bar{\phi}^*}{G_2}\\ 0 & \frac{r}{\sqrt{r^2-2\bar{m}r+\bar{Q}^2}G_2H_1} & 0 & 0\\ 0 & 0 & \frac{r\sin\bar{\phi}^*}{FG_2} & -\frac{rF\sin\theta\cos\bar{\phi}^*}{G_2} \end{pmatrix}, \tag{20}$$

where

$$F = \sqrt{1+2\bar{m}\bar{A}\cos\theta+\bar{Q}^2\bar{A}^2\cos^2\theta}\,,$$
$$G_2 = (\bar{A}r\cos\theta+1)\,, \qquad H_1 = \sqrt{\bar{A}^2r^2-1}\,.$$

Taking $\lim_{\bar{A}\to 0}$ in (20), the associate metric will have the Reissner–Nordström space-time. Now we are going to calculate the energy content of (20). Before we do this let us give a brief review of the derivation of the gravitational energy-momentum.

Multiplication of the symmetric part of (10) by the appropriate inverse tetrad fields yields the result that it is to have the form [35, 47]

$$\partial_\nu\left(-e\Sigma^{a\lambda\nu}\right) = -\frac{ee^{a\mu}}{4}\left(4\Sigma^{b\lambda\nu}T_{b\nu\mu} - \delta^\lambda_\mu\Sigma^{bdc}T_{bcd}\right) - 4\pi e^a{}_\mu T^{\lambda\mu}. \tag{21}$$

By restricting the space-time index λ to assume only spatial values, (21) takes the form [35]

$$\partial_0\left(e\Sigma^{a0j}\right) + \partial_k\left(e\Sigma^{akj}\right) = -\frac{ee^{a\mu}}{4}\left(4\Sigma^{bcj}T_{bc\mu} - \delta^j_\mu\Sigma^{bcd}T_{bcd}\right) - 4\pi ee^a{}_\mu T^{j\mu}. \tag{22}$$

Note that the last two indices of Σ^{abc} and T^{abc} are antisymmetric. Taking the divergence of (22) with respect to j yields

$$- \partial_0 \partial_j \left(-\frac{1}{4\pi} e \Sigma^{a0j} \right) = - \frac{1}{16\pi} \partial_j \left[e e^{a\mu} \left(4 \Sigma^{bcj} T_{bc\mu} - \delta^j_\mu \Sigma^{bcd} T_{bcd} \right) - 16\pi (e e^a{}_\mu T^{j\mu}) \right]. \tag{23}$$

In the Hamiltonian formulation of the TEGR [13–15, 44], the momentum canonically conjugated to the tetrad components e_{aj} is given by

$$\Pi^{aj} = -\frac{1}{4\pi} e \Sigma^{a0j},$$

and the gravitational energy-momentum P^a contained within a volume V of the three-dimensional spacelike hypersurface is defined by [35]

$$P^a = -\int_V \mathrm{d}^3 x \partial_j \Pi^{aj}. \tag{24}$$

If no condition is imposed on the tetrad field, P^a transforms as a vector under the global $SO(3,1)$ group. It describes the gravitational energy-momentum with respect to observers adapted to e^a_μ. These observers are characterized by the velocity field $u^\mu = e^\mu_{(0)}$ and by the acceleration f^μ

$$f^\mu = \frac{\mathrm{D} u^\mu}{\mathrm{d} s} = \frac{\mathrm{D} e^\mu_{(0)}}{\mathrm{d} s} = u^a \nabla_a e^\mu_{(0)}.$$

Let us assume that the space-time is asymptotically flat. The total gravitational energy-momentum is given by

$$P^a = -\oint_{S\to\infty} \mathrm{d} S_k \Pi^{ak}. \tag{25}$$

The field quantities are evaluated on a surface S in the limit $r \to \infty$.

Now we are going to apply (25) to the tetrad field (20) to calculate the energy content. We perform the calculations in Cartesian coordinates. Equations (24) and (25) assumed that the reference space is determined by a set of tetrad fields e^a_μ for the flat space-time, such that the condition $T^a_{\mu\nu} = 0$ is satisfied. It is clear from (22) that the only components which contribute to the energy is $\Sigma^{00\alpha}$. Thus substituting the solution (20) into (7), we obtain the following non-vanishing value:

$$\begin{aligned} \Pi^{0a} &\cong \frac{x^a}{\kappa r^3 (x^2+y^2)\left(1 - \frac{2\bar{m}}{r} + \frac{q^2}{r^2}\right)} \\ &\quad \times \left(r^3 + \left\{ r - 6\bar{m} + \frac{3\bar{Q}^2}{r} \right\} (x^2+y^2) \right), \quad a = 1,2, \\ \Pi^{03} &\cong \frac{z}{\kappa r^3 \left(1 - \frac{2\bar{m}}{r} + \frac{\bar{Q}^2}{r^2}\right)} \left(r - 6\bar{m} + \frac{3\bar{Q}^2}{r} \right). \end{aligned} \tag{26}$$

Substituting from (26) into (25) we get the form of the energy contained within a sphere of radius R as given by

$$\begin{aligned} P^{(0)} = E(R) &= -R \left(1 - \frac{2\bar{m}}{R} + \frac{\bar{Q}^2}{R^2} \right)^{-3/2} \left(1 - \frac{4\bar{m}}{R} + \frac{2\bar{Q}^2}{R^2} \right) \\ &\cong -\left(R - \bar{m} + \frac{\bar{Q}^2}{2R} \right). \end{aligned} \tag{27}$$

5 Regularized expression for the gravitational energy-momentum and localization of energy

An important property of the tetrad fields that satisfy the condition of (3) is that in the flat space-time limit $e^a_\mu(t,x,y,z) = \delta^a_\mu$, and therefore the torsion $T^\lambda_{\mu\nu} = 0$. Hence for the flat space-time it is normal to consider a set of tetrad fields, such that $T^\lambda_{\mu\nu} = 0$ *in any coordinate system*. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (3). Moreover, for such tetrad fields the torsion $T^\lambda_{\mu\nu} \neq 0$ for the flat space-time [47]. It might be argued, therefore, that the expression for the gravitational energy-momentum (24) is restricted to a particular class of tetrad fields, namely, to the class of frames such that $T^\lambda_{\mu\nu} = 0$, if E^a_μ represents the flat space-time tetrad field [47]. To explain this, let us calculate the flat space-time of the tetrad field of (20), which is given by

$$\left(E^a_\mu \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -r\cos\bar{\phi}^* & -r\sin\theta\sin\bar{\phi}^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -r\sin\bar{\phi}^* & r\sin\theta\cos\bar{\phi}^* \end{pmatrix}. \tag{28}$$

Expression (28) yields the following non-vanishing torsion components:

$$\begin{aligned} &T_{(1)21} = -\cos\bar{\phi}^*, && T_{(1)31} = -\sin\theta\cos\bar{\phi}^*, \\ &T_{(1)23} = r\sin\bar{\phi}^*(\cos\theta + 1), && T_{(3)12} = \sin\bar{\phi}^*, \\ &T_{(3)13} = -\sin\theta\sin\bar{\phi}^*, && T_{(3)23} = -r\cos\bar{\phi}^*(\cos(\theta)+1). \end{aligned} \tag{29}$$

The tetrad field (28) when written in Cartesian coordinates will have the form

$$\left(E^a_\mu(t,x,y,z) \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{y^2 r - x^2 z}{r(x^2+y^2)} & -\frac{yx(z+r)}{r(x^2+y^2)} & \frac{x}{r} \\ 0 & \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ 0 & -\frac{yx(z+r)}{r(x^2+y^2)} & \frac{x^2 r - y^2 z}{r(x^2+y^2)} & \frac{y}{r} \end{pmatrix}. \tag{30}$$

In view of the geometric structure of (30), we see that (20) does not display the asymptotic behavior required by (3). Moreover, in general the tetrad field (30) is adapted to accelerated observers [35, 44, 47]. To explain this, let us consider a boost in the x-direction of (30). We find

$$\left(E^a_\mu(t,x,y,z)\right) = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma\dfrac{y^2r-x^2z}{r(x^2+y^2)} & -\dfrac{yx(z+r)}{r(x^2+y^2)} & \dfrac{x}{r} \\ 0 & \dfrac{x}{r} & \dfrac{y}{r} & \dfrac{z}{r} \\ 0 & -\dfrac{yx(z+r)}{r(x^2+y^2)} & \dfrac{x^2r-y^2z}{r(x^2+y^2)} & \dfrac{y}{r}, \end{pmatrix}, \tag{31}$$

where v is the speed of the observer and $\gamma = \sqrt{1-v^2}$. It can be shown that along an observer's trajectory whose velocity is determined by

$$u^u = E^\mu_{(0)} = (\gamma, -v\gamma, 0, 0)\,, \tag{32}$$

the quantities

$$\phi^{(k)}_{(j)} = u^i\left(E^{(k)}_m \partial_i E^m_{(j)}\right),$$

constructed out of (31) are non-vanishing. This fact indicates that along the observer's path the spatial axis $E^\mu_{(a)}$ rotate [44, 47]. In spite of the above problems discussed for the tetrad field of (20) it yields a satisfactory value for the total gravitational energy-momentum, as we will discuss.

In (24) and (25) it is implicitly assumed that the reference space is determined by a set of tetrad fields e^a_μ for flat space-time, such that the condition $T^a_{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time tetrad fields for which $T^a_{\mu\nu} \neq 0$. In this case (24) may be generalized [44, 47] by adding a suitable reference space subtraction term, exactly like in the Brown–York formalism [55–57].

We will denote $T^a_{\mu\nu}(E) = \partial_\mu E^a_\nu - \partial_\nu E^a_\mu$ and $\Pi^{aj}(E)$ as the expression of Π^{aj} constructed out of the flat tetrad E^a_μ. *The regularized form of the gravitational energy-momentum* P^a *is defined by* [44, 47]

$$P^a = -\int_V \mathrm{d}^3x \partial_k [\Pi^{ak}(e) - \Pi^{ak}(E)]\,. \tag{33}$$

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by tetrad fields E^a_μ, obtained from e^a_μ by requiring the vanishing of the physical parameters like mass, angular momentum, etc. Assuming that the space-time is asymptotically flat, then (33) can have the form [44, 47]

$$P^a = -\oint_{S\to\infty} \mathrm{d}S_k [\Pi^{ak}(e) - \Pi^{ak}(E)]\,, \tag{34}$$

where the surface S is established at spacelike infinity. Equation (34) transforms as a vector under the global $SO(3,1)$ group. Now we are in a position to prove that the tetrad field (20) yields a satisfactory value for the total gravitational energy-momentum.

We will integrate (34) over a surface of constant radius $x^1 = r$ and require $r \to \infty$. Therefore, the index k in (34) takes the value $k = 1$. We need to calculate the quantity

$$\Sigma^{(0)01} = e^{(0)}_0 \Sigma^{001} = \frac{1}{2} e^{(0)}_0 (T^{001} - g^{00}T^1)\,.$$

Evaluating the above equation we find

$$-\Pi^{(0)1}(e) = \frac{1}{4\pi} e \Sigma^{(0)01} = -\frac{1}{4\pi} r \sin(\theta) \sqrt{1 - \frac{2\bar{m}}{r} + \frac{\bar{Q}^2}{r^2}}\,, \tag{35}$$

and the expression of $\Pi^{(0)1}(E)$ is obtained by just making $\bar{m} = 0$ and $\bar{Q} = 0$ in (35); it is given by

$$\Pi^{(0)1}(E) = \frac{1}{4\pi} r \sin(\theta)\,. \tag{36}$$

Thus the gravitational energy contained within a sphere of radius R is given by

$$\begin{aligned} P^0 &\cong \int_{r\to R} \mathrm{d}\theta\, \mathrm{d}\phi \frac{1}{4\pi} \sin(\theta) \left\{ -r\left(1 - \frac{\bar{m}}{r} + \frac{\bar{Q}^2}{2r^2}\right) + r \right\} \\ &= \bar{m} - \frac{\bar{Q}^2}{2R}\,, \end{aligned} \tag{37}$$

which is the expected result.

6 Main results and discussion

The main results of this paper are the following:

- Simple expression of the roots of the structure function has been obtained in (16).
- The singularities of the tetrad field of (15) are shown to be related to the roots of the structure function.
- The tetrad field given in (15) with $(t, x, y, \bar{\phi}^*)$ has been transformed to spherical polar coordinates $(t, r, \theta, \bar{\phi}^*)$.
- Setting the physical parameters equal to zero in the tetrad field given in (20), i.e. $\bar{m} = 0$ and $\bar{Q} = 0$, we have obtained a non-Minkowskian space-time.
- It is well known that calculations of the global energies and momenta in TEGR are much easier than in GR. Therefore, we have used the regularized expression of the gravitational energy-momentum given in (34) to calculate the mass-energy given by (37).

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